

the details of the model. We derive exactly the exact expressions for the clustering and degree of a network of small world properties in the whole parameter space. We discuss these networks in the light of the present detailed analysis.

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## ION

has been devoted to of a wide range of so-be described as net-Internet [3, 4, 5, 6] or communities [8], food-networks [10, 11, 12, 13], in which nodes represent and links the physical system, many of these network properties and dynamical properties have been studied for by classical models. In particular, small-world properties [17] (where the clustering coefficient is defined as the number of triangles divided by the number of nodes squared) seem to emerge in many systems. The properties governing the topological properties imply a small average distance and a short average distance has a considerable impact on the dynamics taking place on top of the network. Small-world (SF) networks have been shown to be robust against damage (absence of a central node) [20] and prone to epidemics (above a threshold) [21, 22, 23, 24]. The topological properties of small-world and scale-free networks imply non-trivial degree correlations. Recently, an interest has been introduced by Klemm and

study how clustering affects the resilience to damage in networks [26, 27].

In this paper we study a simple model. We find analytical values of active nodes. In addition, large scale simulations show noticeable variability. In particular, the distribution of  $m$  for the general case of a small-world topology is also susceptible to the construction algorithm. In simulations we study the whole range of parameters. We calculate the clustering and connectivity coefficients and variability with respect to the network. Extensive numerical simulations show a picture presented here.

In the generated small-world properties we find a network of nodes and number of nodes for a given topology is therefore star-shaped graphs. In one dimensional lattices properties. In particular, processes might be heard as the average distance among nodes to a one-dimensional lattice. We discuss the properties of the network.

set of active nodes  $\mathcal{A}$ ,  
 $k$  denotes the in-degree of  
the model is quite sensitive  
3 are performed and,  
the following cases.

*before* step 3.

*after* step 3.

is solved analytically  
on, after introducing  
ve node has in-degree  
order of steps 2 and  
n-degree distribution

(2)

using  $a = m$ . In this  
nversely proportional  
s  $(m + k^{\text{in}})^{-1}$  and the  
be  $P(k) = 2m^2k^{-3}$ .  
n mechanism the net-  
cient that approaches  
limit [25].

n claimed that finite  
ty distribution shows  
vior. We shall see in  
0 the model presents  
which yields a con-  
the  $m \rightarrow \infty$  limit. In  
ology is very sensible  
ving algorithms.

with probability

$$p_d(K) = \frac{1}{1 + \dots}$$

Each time the oldest  
increases by one and  
oldest node has in-  
it is not deactivated  
degree 2, 3, ...  $K -$   
creating a deactivated  
the probability that  
in  $K - 2$  steps and

$$\tilde{P}(K) =$$

$$=$$

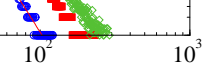
where  $\Gamma(x)$  is the s  
the other hand, eve  
deactivated, the oth  
Hence, in the  $K -$   
generation of a node  
in-degree 1 are crea  
with in-degree 1 cre

$$\tilde{P}_1 = \sum_{K=1}^{\infty}$$

Therefore, the in-de

$$P(k^{\text{in}}) = C$$

where  $C$  is a consta  
condition  $\sum_{k^{\text{in}}} P(k^{\text{i}}$



A for  $a = m$ , network of  $m$ . The continuous  $\gamma = 2$  given by Eq. (10). The exponent  $\gamma$  as a function of  $a$ .

previous expression use to obtain that the in-degree distribution behavior

$$\gamma = 2 + a. \quad (9)$$

and nodes is  $m$  then the out-degree) is  $m + k^{\text{in}}$  shifted by  $m$ . For the degree distribution

$$P(k) \sim k^{-\gamma} \quad (10)$$

$$P(k) \sim k^{-\gamma} \quad (11)$$

distribution obtained from Eq. (9) for  $a = m$ . For  $m = 2$  we find agreement with the data (Fig. 1), with a power law  $\gamma = 4$ . In the limiting case  $a \rightarrow 0$  predicts the exponent  $\gamma = 2$  as a lower bound. Hence,

$$3 < \gamma \leq 4 \quad (11)$$

The probability that a node has degree  $K$  is given by

$$\tilde{P}(K) = \frac{1}{Z} \sum_{k=0}^{\infty} P(k) K^k$$

$$= \frac{1}{Z} \sum_{k=0}^{\infty} P(k) K^k$$

In the process of creating a network of nodes of in-degree  $k$ , the number of nodes with in-degree  $k$  is

$$\tilde{P}_0 = \sum_{K=0}^{\infty} P(K) K^0$$

Thus, the analytic expression for the degree distribution is

$$P(k^{\text{in}}) = C \cdot k^{-\gamma}$$

with the normalization condition

$$C = \frac{1}{\sum_{k=0}^{\infty} k^{-\gamma}}$$

From here follows the expression for the degree distribution (where  $k = m + k^{\text{in}}$ )

$$P(k) = \begin{cases} \frac{\Gamma(2)}{\Gamma(a+1)} & \text{for } k = m \\ \frac{\Gamma(2)}{\Gamma(a+1)} k^{-\gamma} & \text{for } k > m \end{cases}$$

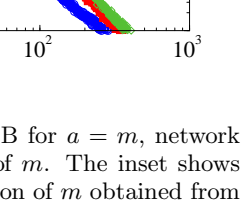


FIG. 3: Scaling of the network model with  $a = m = 1$ . (a)  $P_N(k)$  vs  $N$  for  $k = 1, 2, 3, 4$ . (b)  $P_N(k)^{-1}$  vs  $N$  for  $k = 1, 2, 3, 4$ . The inset shows the scaling of  $m$  obtained from the network model.

show a power law decay continuously increasing with  $m$  such that for  $m < 10$  the scaling differs from the power law.

For  $a = 1$  the analytic solution is readily seen from the power law.

In fact, the solution in the thermodynamic limit of the weight of the nodes with respect to the nodes with degree  $k$  is rooted in the exponent  $\gamma = -2$ , which in the thermodynamic limit, model  $B$  has average degree  $\langle k \rangle = 1$ . This implies that there is a power law decay of the network size  $N$  in the degree  $k$  for  $k = 1$ , dependence that is not seen in the solution since we are in the thermodynamic limit. We can write the general form of the degree distribution  $P_N(k)$  imposed by  $N$  nodes,  $P_N(k) \sim k^{-\gamma}$ , such that there is a cutoff at  $k_c$ . Assuming that the network has the same functional form for  $a = 1$  and  $m = 1$ , we can write the degree distribution as  $P_N(k) \sim k^{-\gamma}$  for  $k < k_c$  and  $P_N(k) \sim k^{-\gamma} e^{-k/k_c}$  for  $k > k_c$ . The cutoff  $k_c$  is determined by the network size  $N$  and the average degree  $\langle k \rangle = 1$ . For  $a = 1$  and  $m = 1$ , the cutoff  $k_c$  is determined by the network size  $N$  and the average degree  $\langle k \rangle = 1$ . For  $a = 1$  and  $m = 1$ , the cutoff  $k_c$  is determined by the network size  $N$  and the average degree  $\langle k \rangle = 1$ .

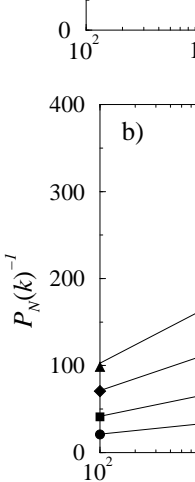


FIG. 3: Scaling of the network model with  $a = m = 1$ . (a)  $P_N(k)$  vs  $N$  for  $k = 1, 2, 3, 4$ . (b)  $P_N(k)^{-1}$  vs  $N$  for  $k = 1, 2, 3, 4$ . The inset shows the scaling of  $m$  obtained from the network model.

approximation that results in the following expression for  $C_1$ :

$$C_1 = 1 - \frac{2 \ln(3)}{\ln \left( \frac{1 + \sqrt{5}}{2} \right)}$$

For finite SF networks with a power law degree distribution  $k^{-\gamma}$ , the maximum degree  $k_c$  is determined by the network size  $N$  and the average degree  $\langle k \rangle = 1$ . For  $a = 1$  and  $m = 1$ , the cutoff  $k_c$  is determined by the network size  $N$  and the average degree  $\langle k \rangle = 1$ . For  $a = 1$  and  $m = 1$ , the cutoff  $k_c$  is determined by the network size  $N$  and the average degree  $\langle k \rangle = 1$ .

$$1 - C_1$$

tion has a divergent

own that the deactivation order in which steps 2 and 3 are performed affects the distributions with a power-law exponent depending on the order. This is rather sensible to the order of variation. This is in contrast to previous works where this was not the case [27], prompting that in those works should

## EFFICIENT

tribution and compute the clustering coefficient of the node  $i$ . For a given  $a$  we can perform an analytical solution of  $a$  and  $m$  and for a given  $m$  we can compute the clustering coefficient of the network as undirected graph. The clustering coefficient of the node

node  $i$  is defined by

$$(24)$$

between the neighbors of node  $i$ . The maximum possible value of  $c(k)$  in this model new edges are added between node  $i$  and the added node. The clustering coefficient of active nodes and inactive nodes is different. Moreover, all the active nodes have the same clustering coefficient. When we add a node to the network, the clustering coefficient increases by one and

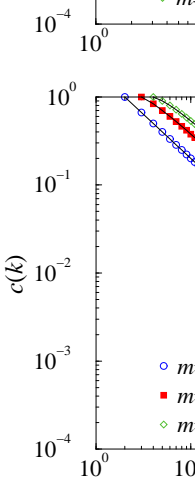


FIG. 4: Clustering coefficient for different values of  $m$ . Numerical simulations of the network for network size  $N = 10^5$  and the analytical solution.

$i$  was created. Besides the degree  $m$ , thus  $e_i(0) = 1$ . Integrating Eq. (24) and substituting the result that  $t = k_i - m$ , we

$$\begin{aligned} c(k) &= \frac{m}{k} \\ &= \frac{2(m-1)}{k} \end{aligned}$$

where the last expression is obtained after some algebraic manipulation.

s of the neighbors of the quantity  $k_{nn,i}$  does degree of the node  $i$ . relations are present. the degree of the node ed. In particular, we ation. In the first sit- ty will connect more s; a property referred e opposite side, it is ng”; i.e. highly con- ed to nodes with low

n the node is added , where  $\langle k \rangle_{\mathcal{A}}$  is the rs. Then, if the node eighbor of the  $m - 1$  y time a new node is creases by  $(m - 1) +$  the remaining  $m - 1$  e and the  $m$  because nce

$$), \quad (28)$$

this equation, taking  $D_i(0) = m \langle k \rangle_{\mathcal{A}}$  and hat

$$+ m \langle k \rangle_{\mathcal{A}} \quad (29)$$

Now, when an active remains fixed but the will still increase until he infinite time limit,

$$(30)$$

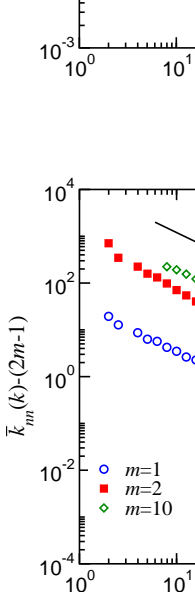
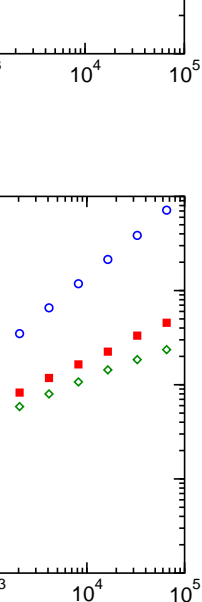


FIG. 5: Average near degree  $k$  for different from numerical simul up to a network size tions. The continuous dependency  $\bar{k}_{nn}(k) -$

time average of  $\langle k \rangle$  nectivity of any dea ( $m = 2$  for model . gree of an active no the degree of the re remaining nodes have with  $m = 2$ , and d dependently of the Therefore in this c



ee as a function of the  
f  $m$ . The points were  
f (a) model A and (b)  
<sup>5</sup>, averaging over 1000

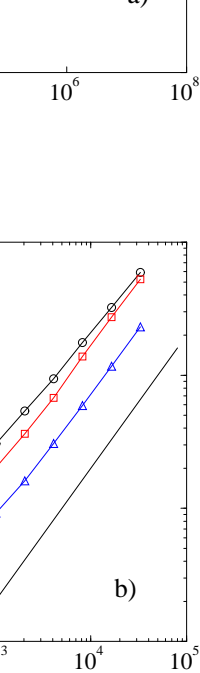
$1/k$  even for  $m \neq 2$ ,  
the case of model B,  
 $m = 1$  and  $m = 10$   
values of  $m$ . Thus,  
n the active node de-  
for intermediate val-  
we find that correla-  
e of “disassortative”  
s are preferably con-  
es. It is also worth

limit  $\gamma = 3$  for  $m$   
mically with  $N$ . O  
 $2 \leq \gamma < 3$ ,  $\langle \bar{k}_{nn} \rangle_N$   
law. This implies th  
and that in the the  
form first the limit  $N$   
connectivity curve  
larger values. This  
nearest neighbor cor  
quantity since the  $k$   
after the  $N \rightarrow \infty$  li  
 $N$  is related to a ge  
diverging connectivi  
the detailed balance

## VI. DIAMETER

Another fundame  
networks is identifi  
length among node  
minimum path betw  
imum number of in  
versed to go from n  
path length  $\langle d \rangle$  is th  
tance averaged over  
network. Similarly,  
the largest among t  
nodes in the network

While regular net  
tices) have a diam  
inverse of the Eucli  
works show striking  
erage one can go fr  
the system by pass  
intermediate nodes



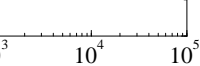
and the average shortest path length  $\langle l \rangle$  for different values of  $m$ . The slope of the curve is approximately 1/3, indicating a linear topology. For clarity, the curve is shown for a scaling factor 5.

implies that the topological properties of the network are approaching those of a linear topology. As  $m$  increases, the deactivation properties of the network are also approaching those of a linear topology.

In Fig. 8 the illustration of the deactivation model B and  $a = m = 3$ . The network is shown as a series of clusters of nodes connected in a chain-like structure, illustrating the linear topology.

FIG. 8: Illustration of deactivation model B. The linear topology is evident.





random walker on the  $d$ -dimensional lattice,  $V = 10^5$  nodes, as well as  $d = 9$  with 6301 nodes.

We will discuss in the last section how these properties might have been affected by the properties of the network.

## CONCLUSIONS

We have provided a detailed analysis of the spreading process introduced in Ref. [25]. The model is being very sensible to the parameters of the model and slight changes can lead to different results. The most striking feature is that the spreading is depending on the number of highly active nodes  $m$  and not on the total degree. When the deactivation probability  $p$  approaches the value  $p_c$ , the spreading properties of networks with  $1 \leq m \leq 10$ . Along with previous works, we have found that the spreading degree correlation and marked disassortative networks link the spreading process. The analytical expression for the spreading is obtained and restricted to the case of a random walker. Interestingly, the SF and the spreading process are

the percolation transition. The spreading process are extremely robust to the changes in the network. A natural question is to study the spreading properties of SF networks. In general, the results obtained in this paper, for this reason, several effects of such correlations on the spreading process on these networks. In Ref. [26] it has been shown that there is a threshold in the case of a random walker.

The presence of a small number of active nodes in the model has been traced back to the finite connectivity of the lattice. In this work we have shown here that the spreading process is dominated by the connectivity in the network. What appears to be the spreading properties of spreading on a small world network structure, with a small number of active nodes. In a coarse grained view, the spreading is dominated by the connectivity of the network. In order to compare the spreading on a standard random network, in Fig. 9 we plot the mean square displacement of a random walker,  $\langle R^2(t) \rangle$ , in brackets denote an average over a random walk on 2500 nodes. In a diffusive system, as we would expect on a lattice, we would expect the spreading to be diffusive behavior, with  $\langle R^2(t) \rangle^{1/2} \sim t^{0.4}$  on the deactivation probability. The spreading is expected from its analysis of spreading and the spreading process cannot, therefore,

by the introduction of  
the usual absence of  
perspective, it would be

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